Problem 1

1. Suppose that $\sum_{n=0}^{\infty} a_n^2$ converges. Must it also be the case that $\sum_{n=0}^{\infty} a_n$ converges?
2. Suppose that $\sum_{n=0}^{\infty} a_n$ converges. Must it also be the case that $\sum_{n=0}^{\infty} a_n^2$ converges?

Problem 2

Suppose that (a_n) is a sequence of *positive* real numbers such that $\sum_{n=1}^{\infty} a_n$ converges. Let $r_n = \sum_{m=n}^{\infty} a_m$. Prove that if $m < n$,

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

and deduce that $\sum_{n=1}^{\infty} \frac{a_n}{r_n}$ diverges.

Problem 3

Using the identity for geometric series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

for $|x| < 1$, express the following series as expressions not involving summations, and state for which x they converge.

1. $\sum_{k=0}^{\infty} (-1)^k x^{2k}$
2. $\sum_{k=0}^{\infty} (1-x)^{3k}$
3. $\sum_{k=0}^{\infty} \frac{1}{(x-2)^k}$

Problem 4

Prove that if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|$.